

ROTATIONALLY SYMMETRIC HARMONIC DIFFEOMORPHISMS IN PLANE

ZHAN-QING WANG

Department of Mathematics
Jinan University
Guangzhou, 510632
P. R. China
e-mail: 945333067@qq.com

Abstract

In this paper, we study that the existence of rotationally symmetric harmonic diffeomorphism between the punctured complex plane with hyperbolic metric or Euclidean metric.

1. Introduction

The existence or nonexistence of harmonic diffeomorphisms between Riemannian manifolds was studied by many people, see example [1-4, 6-13] and the references therein. In particular, in [6, 10, 12, 13], the authors therein studied the rotational symmetry case. The corresponding questions of Euclidean metric is related to the Nitsche conjecture, see, for example, [5, 14]. Recently, [3, 4, 9] focus on the question of the existence of harmonic diffeomorphism between Riemannian surfaces of annular topological. For example, in [4], Chen et al. proved some necessary and conditions for existence of rotationally symmetric harmonic diffeomorphism between the annuli with the Poincaré metric or Euclidean metric.

2010 Mathematics Subject Classification: Primary 58E20; Secondary 34B15.

Keywords and phrases: harmonic map, rotational symmetry.

Received April 25, 2014

In this notes, using similar methods of [3, 4], we are able to prove that the existence of rotationally symmetric harmonic diffeomorphism between the punctured complex plane with the hyperbolic metric or Euclidean metric. More precisely, let us denote

$$P(a) = \mathbb{D} \setminus \{|z| \leq a, a > 0\} \text{ and } \mathbb{C}_b^* = \mathbb{C} \setminus \{|z| \leq b, b > 0\},$$

where \mathbb{D} is the unit disc and z is the complex coordinate of \mathbb{C} .

Now we state our the main results of this paper.

Theorem 1.1. *For any $b > 0, d > 0$, there exists rotationally symmetric harmonic diffeomorphism from \mathbb{C}_b^* to \mathbb{C}_d^* with its hyperbolic metric.*

Theorem 1.2. *For any $b > 0, 0 < a < 1$, there is no rotationally symmetric harmonic diffeomorphism from \mathbb{C}_b^* onto $P(a)$ with the Poincaré metric; on the other hand, there exists rotationally symmetric harmonic diffeomorphism from $P(a)$ onto \mathbb{C}_b^* with its hyperbolic metric.*

At the same time, we will also consider the Euclidean case, and will prove the following theorems:

Theorem 1.3. *For any $b > 0, d > 0$, there exists rotationally symmetric harmonic diffeomorphism from \mathbb{C}_b^* onto \mathbb{C}_d^* with its Euclidean metric.*

Theorem 1.4. *For any $b > 0, 0 < a < 1$, there is no rotationally symmetric harmonic diffeomorphism from \mathbb{C}_b^* onto $P(a)$ with its Euclidean metric. The converse implications are true.*

The organization of this paper is as follows. In Section 2, we will prove Theorems 1.1 and 1.2. Theorems 1.3 and 1.4 will be proved at the last section.

2. Case of the Non-Euclidean Metric

For convenience, let us recall the definition about the harmonic maps between surfaces. Let M and N be two oriented surfaces with metrics $\tau^2|dz|^2$ and $\sigma^2|du|^2$, respectively, where z and u are local complex coordinates of M and N , respectively. A C^2 map u from M to N is harmonic if and only if u satisfies

$$u_{z\bar{z}} + \frac{2\sigma_u}{\sigma} u_z u_{\bar{z}} = 0. \quad (2.1)$$

Our first goal is to show that Theorem 1.1.

Proof of Theorem 1.1. First of all, let us denote (r, θ) as the polar coordinates of \mathbb{C}_b^* , and u as the complex coordinates of \mathbb{C}_d^* in \mathbb{C} . The hyperbolic metric $\sigma_1|du|$ on \mathbb{C}_d^* can be written as

$$\frac{2d|du|}{|u|^2 - d^2}, \quad (2.2)$$

where $|u|$ is the norm of u with respect to the Euclidean metric.

Given u is a rotationally symmetric harmonic diffeomorphism from \mathbb{C}_b^* onto \mathbb{C}_d^* with the metric $\sigma_1|du|$. Because \mathbb{C}_b^* , \mathbb{C}_d^* and the metric $\sigma_1|du|$ are rotationally symmetric, we can assume that such a map u has the form $u = f(r)e^{i\theta}$. By substituting u , σ_1 into (2.1), we can get

$$f'' + \frac{f'}{r} - \frac{f}{r^2} - \frac{2f}{f^2 - d^2} \left[(f')^2 - \frac{f^2}{r^2} \right] = 0 \text{ for } r > b. \quad (2.3)$$

Since u is a harmonic diffeomorphism from \mathbb{C}_b^* onto \mathbb{C}_d^* , we have

$$f(b) = d, f(+\infty) = +\infty \text{ and } f'(r) > 0 \text{ for } r > b, \quad (2.4)$$

or

$$f(b) = +\infty, f(+\infty) = d \text{ and } f'(r) < 0 \text{ for } r > b. \quad (2.5)$$

Regarding r as a function of f , we have the following relations:

$$f_r = r_f^{-1}, \quad f_{rr} = -r_f^{-3} r_{ff}. \quad (2.6)$$

Combining (2.3) with (2.6), we get

$$\frac{r''}{r} - \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 f + \frac{2f}{f^2 - d^2} \left[\frac{r'}{r} - f^2 \left(\frac{r'}{r}\right)^3 \right] = 0. \quad (2.7)$$

Let $x = (\ln r)'(f)$, by (2.7), we obtain that

$$(f^2 - d^2)x' + 2fx - x^3(fd^2 + f^3) = 0.$$

Solving the above Bernoulli equation, one gets

$$x^{-2} = f^2 + c_0(f^2 - d^2)^2, \quad (2.8)$$

for some nonnegative constant c_0 depending on the choice of the function f . Therefore,

$$x = \frac{1}{\sqrt{f^2 + c_0(f^2 - d^2)^2}}.$$

For the case $c_0 = 0$, we have

$$r = k_0 f, \quad (2.9)$$

where k_0 is some constant. Solving Equation (2.9) with (2.4), we have

$$f = \frac{d}{b} r, \quad (2.10)$$

and

$$f'(r) = \frac{d}{b} > 0.$$

Equation (2.9) contradicts to the boundary condition (2.5).

Now we consider the case $c_0 > 0$, we get

$$\int_r^{+\infty} d \ln t = \int_f^{+\infty} \frac{1}{\sqrt{t^2 + c_0(t^2 - d^2)^2}} dt. \quad (2.11)$$

For $r > 0$, the improper integral of Equation (2.11) on the right is the convergence and on the left is divergent. This leads to a contradiction. We completed the proof of Theorem 1.1. \square

Now we want to prove Theorem 1.2.

Proof of Theorem 1.2. Let us begin to show the first part of this theorem. The idea is similar to the proof of Theorem 1.1 in [4], so we just sketch the proof here. Using (2.7) in [4], we have

$$x^{-2} = f^2 + c_1(1 - f^2)^2, \quad (2.12)$$

with the boundary condition

$$f(b) = a, f(+\infty) = 1 \text{ and } f'(r) > 0 \text{ for } r > b, \quad (2.13)$$

or

$$f(b) = 1, f(+\infty) = a \text{ and } f'(r) < 0 \text{ for } r > b. \quad (2.14)$$

The proof is by contradiction. Under the condition (2.13), suppose there exist a function $f(r)$ satisfies (2.12), then similar to (2.11), we can get

$$\int_r^{+\infty} d \ln t = \int_f^1 \frac{1}{\sqrt{t^2 + c_1(1 - t^2)^2}} dt.$$

Note that the left integral is divergent for any $r > 0$, so we cannot find a function $f(r)$ such that $f' > 0$. This contradicts to the assumption of (2.13). Hence such a rotationally symmetric harmonic diffeomorphism with the boundary condition (2.13) does not exist. Similarly, one can show that there is no rotationally symmetric harmonic diffeomorphism with the boundary condition (2.14). Therefore, the first part of Theorem 1.2 holds.

Now let us prove the second part of this theorem. From the proof of Theorem 1.1, it suffices for us to show that the equation

$$x^{-2} = f^2 + c_2(f^2 - b^2)^2, \quad (2.15)$$

with the boundary condition

$$f(a) = b, f(1) = +\infty \text{ and } f'(r) > 0 \text{ for } a < r < 1, \quad (2.16)$$

or

$$f(a) = +\infty, f(1) = b \text{ and } f'(r) < 0 \text{ for } a < r < 1. \quad (2.17)$$

Firstly, consider the $c_2 = 0$. By Equation (2.15), we get

$$r = k_1 f, \quad (2.18)$$

for k_1 to denote a generic constant. Combining Equation (2.18) with (2.16) or (2.17), we easy to see Equation (2.18) has no solution.

Now we consider $c_2 > 0$, then the improper integral $\int_b^{+\infty} \frac{1}{\sqrt{f^2 + c_2(f^2 - b^2)^2}} df$ is convergent for any $b > 0$. In this case, let us consider the harmonic diffeomorphism with the boundary condition (2.16), which say $f(a) = b, f(1) = +\infty$. By solving Equation (2.15), we can get

$$\int_a^1 d \ln r = \int_b^{+\infty} \frac{1}{\sqrt{f^2 + c_2(f^2 - b^2)^2}} df. \quad (2.19)$$

Similarly, combining Equation (2.15) with (2.17), we have

$$\int_a^1 d \ln r = -\int_{+\infty}^b \frac{1}{\sqrt{f^2 + c_2(f^2 - b^2)^2}} df = \int_b^{+\infty} \frac{1}{\sqrt{f^2 + c_2(f^2 - b^2)^2}} df.$$

Note that the right improper integral of the Equation (2.19) is divergent for any $c_2 < 0, 1 > r > a$. Hence any rotationally symmetric harmonic diffeomorphism from $P(a)$ onto \mathbb{C}_d^* should satisfy (2.19) for some $c_2 > 0$. Let

$$g(c_2) = \int_a^1 d \ln r - \int_b^{+\infty} \frac{1}{\sqrt{f^2 + c_2(f^2 - b^2)^2}} df.$$

It is easy to see that $g(c_2)$ is a monotonically increasing continuous function with

$$\lim_{c_2 \rightarrow +\infty} g = -\ln a > 0 \quad \text{and} \quad \lim_{c_2 \rightarrow 0} g = -\infty < 0.$$

Therefore, there exists c_2 such that $g(c_2) = 0$.

The proof of the second part of Theorem 1.2 is now complete. □

3. Case of the Euclidean Metric

We are going to prove Theorem 1.3.

Proof of Theorem 1.3. Let us prove this theorem. The ideal is similar to the proof of Theorem 1.3 in [3], so we just sketch the proof here. Suppose there is such a harmonic diffeomorphism φ from \mathbb{C}_b^* onto \mathbb{C}_d^* with its Euclidean metric with the form $\varphi = h(r)e^{i\theta}$, then we can get

$$h'' + \frac{h'}{r} - \frac{h}{r^2} = 0 \text{ for } r > b > 0, \quad (3.1)$$

with the boundary condition

$$h(b) = d, h(+\infty) = +\infty \text{ and } h'(r) > 0 \text{ for } r > b > 0, \quad (3.2)$$

or

$$h(b) = +\infty, h(+\infty) = d \text{ and } h'(r) < 0 \text{ for } r > b > 0. \quad (3.3)$$

Solving (3.1), one get

$$h(r) = \frac{c_3}{r} + c_4 r, \quad (3.4)$$

where c_3 and c_4 are constants. Clearly, combining (3.4) with (3.3) have no solution. By Equation (3.4) and condition (3.2), we can get

$$c_3 = bd - c_4 b^2, \quad (3.5)$$

and from $h'(r) > 0$, we get

$$c_3 < \frac{bd}{2} \text{ and } c_4 > \frac{d}{2b}. \quad (3.6)$$

Provided that Equations (3.5) and (3.6), there exists the equation h . We have thus proved Theorem 1.3. \square

Proof of Theorem 1.4. Let us begin to show the first part of this theorem, i.e., the nonexistence of rotationally symmetric harmonic diffeomorphism from \mathbb{C}_b^* onto $P(a)$ with its Euclidean metric. This theorem can be proved by the same method as employed in the last theorem, it suffices for us to show that there is no function h such that

$$h(r) = \frac{c_3}{r} + c_4 r, \quad (3.7)$$

with the boundary condition

$$h(b) = a, h(+\infty) = 1 \text{ and } h'(r) > 0 \text{ for } r > b > 0, \quad (3.8)$$

or

$$h(b) = 1, h(+\infty) = a \text{ and } h'(r) < 0 \text{ for } r > b > 0, \quad (3.9)$$

for suitable constants c_3 and c_4 . From (3.7), we know that $h(+\infty) = 0$ or $\pm\infty$ which contradicts the boundary condition. So the first part of this theorem holds.

Now let us prove the second part of this theorem, that is, show the nonexistence of rotationally symmetric harmonic diffeomorphism from $P(a)$ onto \mathbb{C}_b^* with its Euclidean metric. Similar to the proof of the first part, it suffices for us to show that there is no function h such that

$$h(r) = \frac{c_3}{r} + c_4 r, \quad (3.10)$$

with the boundary condition

$$h(a) = b, h(1) = +\infty \text{ and } h'(r) > 0 \text{ for } a < r < 1, \quad (3.11)$$

or

$$h(a) = +\infty, h(1) = b \text{ and } h'(r) < 0 \text{ for } a < r < 1. \quad (3.12)$$

Clearly, (3.10) guarantees $h(1)$ and $h(a)$ are finite. This contradicts the conditions (3.11) and (3.12). Hence such a function h does not exist, the second part of Theorem 1.4 has been proved. \square

Acknowledgement

My deepest gratitude to Professor Xu-Qian Fan, for his useful discussion.

References

- [1] K. Akutagawa, Harmonic diffeomorphisms of the hyperbolic plane, Trans. Amer. Math. Soc. 342(1) (1994), 325-342.
- [2] K. Akutagawa and S. Nishikawa, The Gauss map and spacelike surfaces with prescribed mean curvature in Minkowski 3-space, Tohoku Math. J. (2) 42(1) (1990), 67-82.
- [3] L. Chen, S.-Z. Du and X.-Q. Fan, Rotationally symmetric harmonic diffeomorphisms between surfaces, Abstr. Appl. Anal. (2013), Article ID 512383.
<http://dx.doi.org/10.1155/2013/512383>
- [4] L. Chen, S.-Z. Du and X.-Q. Fan, Harmonic diffeomorphisms between the annuli with rotational symmetry, Nonlinear Anal. 101 (2014), 144-150; Addendum to "Harmonic diffeomorphisms between the annuli with rotational symmetry" [Nonlinear Anal. 101 (2014), 144-150], Nonlinear Anal. 105 (2014), 1-2.
- [5] Johannes C. C. Nitsche, On the module of doubly-connected regions under harmonic mappings, Amer. Math. Soc. Monthly 69(8) (1962), 781-782.
- [6] L. F. Cheung and C. K. Law, An initial value approach to rotationally symmetric harmonic maps, J. Math. Anal. Appl. 289(1) (2004), 1-13.
- [7] F. Hélein, Harmonic diffeomorphisms with rotational symmetry, J. Reine Angew. Math. 414 (1991), 45-49.

- [8] D. Kalaj, On the Nitsche conjecture for harmonic mappings in \mathbb{R}^2 and \mathbb{R}^3 , Israel J. Math. 150 (2005), 241-251.
- [9] M. Leguil and H. Rosenberg, On harmonic diffeomorphisms from conformal annuli to Riemannian annuli, Preprint.
- [10] A. Ratto and M. Rigoli, On the asymptotic behaviour of rotationally symmetric harmonic maps, J. Differential Equations 101(1) (1993), 15-27.
- [11] R. M. Schoen, The role of harmonic mappings in rigidity and deformation problems, Complex Geometry (Osaka, 1990), 179-200; Lecture Notes in Pure and Appl. Math. 143, Dekker, New York, 1993.
- [12] A. Tachikawa, Rotationally symmetric harmonic maps from a ball into a warped product manifold, Manuscripta Math. 53(3) (1985), 235-254.
- [13] A. Tachikawa, A nonexistence result for harmonic mappings from \mathbb{R}^n into \mathbb{H}^n , Tokyo J. Math. 11(2) (1988), 311-316.
- [14] T. Iwaniec, L. V. Kovalev and J. Onninen, The Nitsche conjecture, J. Amer. Math. Soc. 24(2) (2011), 345-373.

